Math 113 Expository Paper
Quaternions

Kamyar Salahi

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A *quaternion* is a number of the form

$$a + b\mathbf{i} + c\mathbf{j} + d\mathbf{k}$$

where $a, b, c, d \in \mathbb{R}$. The set of all quaternions is denoted as $\mathbb{H}$. Here $\mathbf{1}$, $\mathbf{i}$, $\mathbf{j}$, and $\mathbf{k}$ are related to one another as follows:

$$\mathbf{i}^2 = \mathbf{j}^2 = \mathbf{k}^2 = -\mathbf{1}$$

$$\mathbf{ij} = -\mathbf{ji} = \mathbf{k}$$

$$\mathbf{jk} = -\mathbf{jk} = \mathbf{i}$$

$$\mathbf{ki} = -\mathbf{ik} = \mathbf{j}$$

Quaternions can be interpreted as either a four-dimensional vector, a scalar and a three-dimensional vector, or an extension of the complex numbers. In the second interpretation, the scalar part is represented by a and the vector part by $b\mathbf{i} + c\mathbf{j} + d\mathbf{k}$.

Addition of two quaternions $\mathbf{x}_1 = a_1 + b_1\mathbf{i} + c_1\mathbf{j} + d_1\mathbf{k}$ and $\mathbf{x}_2 = a_2 + b_2\mathbf{i} + c_2\mathbf{j} + d_2\mathbf{k}$ is defined as follows:

$$\mathbf{x}_1 + \mathbf{x}_2 = (a_1 + a_2) + (b_1 + b_2)\mathbf{i} + (c_1 + c_2)\mathbf{j} + (d_1 + d_2)\mathbf{k}$$

Following the previous relations of $\mathbf{1}$, $\mathbf{i}$, $\mathbf{j}$, and $\mathbf{k}$, the multiplication of two quaternions (Hamiltonian Product) is defined as follows:

$$\begin{aligned} \mathbf{x}_1\mathbf{x}_2 &= (a_1a_2 - b_2b_2 - c_1c_2 - d_1d_2) \\ &+ (a_1b_2 + b_1a_2 + c_1d_2 - d_1c_2)\mathbf{i} \\ &+ (a_1c_2 - b_1d_2 + c_1a_2 - d_1b_2)\mathbf{j} \\ &+ (a_1d_2 + b_1c_2 - c_1b_2 + d_1a_2)\mathbf{k} \end{aligned}$$

Note: In the third interpretation, quaternions can be defined as an ordered pair of two complex numbers (x, y) for $x, y \in \mathbb{C}$. In this definition of quaternions, addition of two quaternions would be defined as $(x_1, y_1) + (x_2, y_2) = (x_1 + x_2, y_1 + y_2)$ and multiplication of two quaternions would be defined as $(x_1, y_1)(x_2, y_2) = (x_1x_2 - \overline{y_1}y_2, \overline{x_1}y_2 + y_1x_2)$.

Quaternions have an identity element $\mathbf{1}$ with the multiplication operation being associative but not commutative. Every quaternion has a non-zero multiplicative inverse defined as follows:

$$(a + b\mathbf{i} + c\mathbf{j} + d\mathbf{k})^{-1} = \frac{1}{a^2 + b^2 + c^2 + d^2}(a - b\mathbf{i} - c\mathbf{j} - d\mathbf{k})$$

As described above, quaternions are a division ring (and are not a field since the multiplication operation is not commutative). A *division ring* is a ring where every non-zero element of the ring has a unique non-zero inverse. In other words, every non-zero element of the ring is a unit.

Now that we have discussed what quaternions are, we will consider how they can be used to represent 3D rotations. One way to describe a 3D rotation (following from Euler's rotation theorem) is by the axis on which the rotation occurs and the magnitude (angle) of the rotation. This axis can be described by a 3-dimensional vector (of unit norm) while the magnitude can be defined by a scalar quantity. Following our second interpretation of quaternions, this axis-angle approach to representing a rotation aligns well with our construction of quaternions. However, we must determine a means of constructing our quaternions such that we may use them to calculate rotations.

Let (x, y, z) be the axis of our rotation and θ be the angle. We define our quaternion

$$\mathbf{q} = \cos(\theta/2) + x\sin(\theta/2)\mathbf{i} + y\sin(\theta/2)\mathbf{j} + z\sin(\theta/2)\mathbf{k}.$$

Notice that

$$|\mathbf{q}|^2 = \cos(\theta/2)^2 + (x^2 + y^2 + z^2)\sin(\theta/2)^2.$$

Since the axis is of unit norm, $x^2 + y^2 + z^2 = 1$. This means that $|\mathbf{q}|^2 = 1$.

We take a 3D vector (a, b, c) and represent it as a purely imaginary quaternion $\mathbf{p} = 0 + a\mathbf{i} + b\mathbf{j} + c\mathbf{k}$. It turns out that conjugating $\mathbf{p}$ with $\mathbf{q}$, $\mathbf{qpq}^{-1}$, yields a result that is equivalent to rotation of the vector (a, b, c) through an angle of θ about the axis (x, y, z). [1] Since the Hamiltonian product preserves the norm, it follows that the norm $|\mathbf{qpq}^{-1}|$ is equivalent to $|\mathbf{p}|$. Here, the fact that $\mathbf{q}$ is of unit norm is crucial here to ensure that the operation preserves lengths and therefore acts consistently with a rotation. One can compose two unit quaternions $\mathbf{q}_1$ and $\mathbf{q}_2$ by simply calculating the Hamiltonian product $\mathbf{q}_1\mathbf{q}_2$. This follows since $\mathbf{q}_1\mathbf{q}_2\mathbf{p}(\mathbf{q}_1\mathbf{q}_2)^{-1} = \mathbf{q}_1\mathbf{q}_2\mathbf{p}\mathbf{q}_2^{-1}\mathbf{q}_1^{-1}$. This composition is still a rotation (the Hamiltonian product of two unit quaternions is a unit quaternion) and is in fact equivalent to the composition of the two corresponding rotations for the unit quaternions.

There are many approaches that are used to represent 3D rotations of vectors including 3×3 orthonormal matrices, axis-angle, Euler angles, and quaternions. Quaternions have some advantage in terms of memory and runtime overhead. Representing an orthonormal matrix requires the storage of 9 numbers. In contrast, quaternions can be stored as a four-tuple. Composition of orthonormal matrices requires 27 multiplications and 18 additions. For quaternions, composition of rotations can be very straightforward since it is simply a multiplication of the corresponding quaternions, using only 16 multiplications and 12 additions. Finally, when solving for an optimization problem involving rotations, it is much easier to use quaternions. This is because the unit-norm constraint of quaternions is easy to enforce. For orthonormal matrices, valid matrices must be both orthonormal and have a determinant of one, yielding seven non-linear constraints on the elements of the matrix. These constraints make optimization using an orthonormal matrix representation unnecessarily challenging.

[1] https://graphics.stanford.edu/courses/cs348a-17-winter/Papers/quaternion.pdf#page=5